DIFFERENTIAL MANIFOLDS HW 3

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1. Exercise 1.14

Employing the summation convention, we have:

$$[u, v]^i = \frac{\partial u^i}{\partial x^j} v^j - \frac{\partial v^i}{\partial x^j} u^j$$

So that:

$$\begin{aligned}
[w, [u, v]]^i &= \frac{\partial w^i}{\partial x^k}[u, v]^k - \frac{\partial^2 u^i}{\partial x^j \partial x^k} w^k v^j - \frac{\partial u^i}{\partial x^j}\frac{\partial v^j}{\partial x^k} w^k \\
&+ \frac{\partial^2 v^i}{\partial x^j \partial x^k} w^k u^j + \frac{\partial v^i}{\partial x^j}\frac{\partial u^j}{\partial x^k} w^k \\
&= \frac{\partial w^i}{\partial x^k}[u, v]^k - \frac{\partial^2 u^i}{\partial x^j \partial x^k} w^k v^j + \frac{\partial^2 v^i}{\partial x^j \partial x^k} w^k u^j
\end{aligned}$$

Taking cyclic permutations of $\{u,\ v,\ w\}$, we easily find:

$$[v, [w, u]]^i = \frac{\partial v^i}{\partial x^k}[w, u]^k - \frac{\partial^2 w^i}{\partial x^j \partial x^k} v^k u^j + \frac{\partial^2 u^i}{\partial x^j \partial x^k} v^k w^j \tag{1.1}$$

$$[u, [v, w]]^i = \frac{\partial u^i}{\partial x^k}[v, w]^k - \frac{\partial^2 v^i}{\partial x^j \partial x^k} u^k w^j + \frac{\partial^2 w^i}{\partial x^j \partial x^k} u^k v^j \tag{1.2}$$

From here it is simple to see that $\frac{\partial w^i}{\partial x^k}[u, v]^k + \frac{\partial v^i}{\partial x^k}[w, u]^k + \frac{\partial u^i}{\partial x^k}[v, w]^k = 0$. Likewise, comparing the second order terms in the above equations, we see that these must all cancel out as well by renaming indices as necessary. Hence, we obtain:

Date: September 3, 2017.

$$[u,[v,w]]+[w,[u,v]]+[v,[w,u]]=0$$

As desired.

2. Exercise 1.23

First check that σ is closed:

(2.1)
$$\begin{aligned}
\mathrm{d}\sigma &= \Big(\frac{-2x^2+y^2+z^2}{r^5}\mathrm{d}x+\frac{-3xy}{r^5}\mathrm{d}y+\frac{-3zx}{r^5}\mathrm{d}z\Big)\wedge\mathrm{d}y\wedge\mathrm{d}z \\
&+\frac{-3xy}{r^5}\mathrm{d}x+\frac{-2y^2+x^2+z^2}{r^5}\mathrm{d}y+\frac{-3zy}{r^5}\mathrm{d}z\Big)\wedge\mathrm{d}z\wedge\mathrm{d}x \\
&+\frac{-3xz}{r^5}\mathrm{d}x+\frac{-3yz}{r^5}\mathrm{d}y+\frac{-2z^2+x^2+y^2}{r^5}\mathrm{d}z\Big)\wedge\mathrm{d}x\wedge\mathrm{d}y \\
&=\frac{-2x^2+y^2+z^2}{r^5}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z+\frac{-2y^2+x^2+z^2}{r^5}\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x \\
&+\frac{-2z^2+x^2+y^2}{r^5}\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y \\
&=\frac{-2r^2+2r^2}{r^5}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z=0
\end{aligned}$$

Following the hint, compute $\mathrm{d}(z\omega/r)$, where ω is given in Exercise 1.22:

(2.2)
$$\begin{aligned}
\mathrm{d}(z\omega/r) &= \frac{1}{\rho^2r^3}\Big(x^2z\mathrm{d}x\wedge\mathrm{d}y-y^2z\mathrm{d}y\wedge\mathrm{d}x+\rho^2x\mathrm{d}z\wedge\mathrm{d}y+\rho^2y\mathrm{d}z\wedge\mathrm{d}x\Big) \\
&=\frac{1}{r^3}(z\mathrm{d}x\wedge\mathrm{d}y+x\mathrm{d}z\wedge\mathrm{d}y+y\mathrm{d}z\wedge\mathrm{d}x)=\sigma
\end{aligned}$$

Hence we can guess our antiderivative to be $z\omega/r$. But this form is singular all along the z axis, so we can split this into the cases for $z<0$ and $z>0$. When $z>0$, note that since with $\omega:=\rho^{-2}\alpha$, α defined obviously, then $d(z/r\omega+\omega)=\sigma$. However,

$$z/r\omega + \omega = \frac{z+r}{r}\rho^{-2}\alpha = \frac{\rho^2}{r(z-r)}\rho^{-2}\alpha = \frac{1}{r(z-r)}\alpha := \Omega^-$$

Hence, this is no longer singular along the lower half space, since $z - r \neq 0$ for negative z. For $z > 0$, merely subtract ω instead:

$$z/r\omega - \omega = \frac{z-r}{r}\rho^{-2}\alpha = \frac{\rho^2}{r(z+r)}\rho^{-2}\alpha = \frac{1}{r(z+r)}\alpha := \Omega^+$$

And we see that $z + r \neq 0$ on the upper half space as well. Hence, we see that our antiderivative is $\frac{1}{r(z+r)}\alpha$ along the upper half space and $\frac{1}{r(z-r)}\alpha$ along the lower half space. Thus, we want to find what this form would be as $z \to 0$, ie, along the boundary of these two spaces. Note that $r \to \rho$ as $z \to 0$, and hence, tending to 0 from either the upper or lower half space, we see:

$$\frac{1}{r(z+r)}\alpha \to \frac{1}{\rho^2}\alpha = \omega$$
$$\frac{1}{r(z-r)}\alpha \to \frac{-1}{\rho^2}\alpha = -\omega$$

However, the upper and lower half spaces are both convex and hence contractible and by Poincare's Lemma both forms for the upper and lower half space are exact since we have already proved they are closed. So:

$$\Omega^+ + \mathrm{d}a = \theta^+$$

$$\Omega^- + \mathrm{d}b = \theta^-$$

For some exact 1-forms θ^+ and θ^-, where a, b are some smooth functions. However, this means that on the xy-plane, $\Omega^+ + \mathrm{d}a = \Omega^- + \mathrm{d}b$. Setting $z = 0$, we see that this implies that $2\omega = \mathrm{d}(a - b)$. But this means that ω is exact on $\mathbb{R}^2\backslash\{0\}$, contradicting the result of exercise 1.22 of the previous homework (cf. HW2). Thus σ cannot possibly be exact.

3. Exercise 2.3

Firstly, it is clear that $S_{ii} \equiv \text{id}$. Also, $S_{ji} : U_{i_*} \to U_{j_*}$ by the given condition that $S_{jj} \circ S_{ii} \subset S_{ji}$. Denote by Y_{i_*} the image of U_{i_*} under the quotient of the given equivalence relation. Then, to see that these i_* are bijections merely note that if $(u, i) = (v, i)$ in Y_{i_*}, then $S_{ii}(u) = v$. But since S_{ii} is the identity, we find $v = u$. Hence this is injective. Surjectivity is clear: the preimage of (u, i) is just u.

Now, it is obvious that the domains of our charts are numerical spaces by the condition given in the problem, hence this satisfies the first condition for a manifold structure. The second condition also holds as well: if $(u, i) \in Y$, then we have $i_*(u) = (u, i)$, so the ranges must cover Y.

Finally, we want to check that intersections are compatible. Suppose that $u \in i_*(Y_{i_*} \cap Y_{j_*})$. Then, $u = S_{ji}(v)$ for some $v \in U_{j_*}$. However, this then implies that $i_*(u) = (v, j)$, and taking the preimage, we see that $j_*^{-1} i_*(u) = v$, and this is well defined since we've already shown the quotient maps are bijections.

This is a diffeomorphism since this is clearly the same as finding the image $S_{ij}(u) = v$, and since each S_{ij} is given to be a diffeomorphism, the transition maps are also diffeomorphisms. Thus we have a unique manifold structure on Y.

Conversely, suppose we have an atlas $\{a\}_{a\in A}$, $a : U_a \to X_a$. Then, obviously the domains are numerical spaces for the $S_{ba} := b^{-1} \circ a : U_a \to$

U_b since A is a chart, and each S_{ba} is a diffeomorphism as a composition of diffeomorphisms. The second condition also holds trivially by the above, since the preimage of any equivalence class (u, a) is simply $u \in U_a$ (since $U_{i_*} = U_a$ as defined above).

Finally, for the third condition, we already know that our charts of the original atlas A are compatible and by the work of the first part, $b_*^{-1} a_* = S_{ab}$. Since $a^{-1} \circ b$ must be a diffeomorphism by compatibility of A, the transition maps $b_*^{-1} a_*$ are diffeomorphisms as well, and we are done.